THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A Tutorial 7

1. For $x \in \mathbb{R}$, the floor of x is defined by $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$. Determine the points of continuity of the following functions:

   (a) $f(x) := \lfloor x \rfloor$,

   (b) $h(x) := \lfloor \frac{1}{x} \rfloor$.

2. Give an example for each of the following:

   (a) $f : \mathbb{R} \to \mathbb{R}$ continuous only at one point,

   (b) $f : \mathbb{R} \to \mathbb{R}$ discontinuous everywhere but $|f|$ continuous everywhere,

   (c) $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R} \backslash \mathbb{Q}$ but discontinuous on $\mathbb{Q}$.

3. Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a continuous at a point $c \in A$. Show that for any $\epsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \epsilon$.

4. Let E be a non-empty subset of $\mathbb{R}$. For $x \in \mathbb{R}$, define $f_E(x) = \inf\{|x - y| : y \in E\}$. Show that f_E is well-defined and is Lipschitz (hence continuous) on $\mathbb{R}$.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

6. **(Alternative proof of Location of Roots Theorem)** Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$. Prove that $f(w) = 0$.